Journal of Statistical Physics, Vol. 71, Nos. 1/2, 1993

A Renormalization Group Analysis of Turbulent Transport

Chetan Nayak[1,2]

Received March 10, 1992; final November 2, 1992

The field-theoretic renormalization group is used to derive scaling relations for the transport of passive scalars by an incompressible velocity field with a specified energy spectrum. Results are obtained with the analog of the ε expansion of critical phenomena and compared to exact results which are available for shear flows in two dimensions. A $1/N$ expansion is proposed for the regions in which the ε expansion fails.

KEY WORDS: Renormalization group; turbulence theory; turbulent transport.

1. INTRODUCTION

The renormalization group has facilitated the derivation of universal scaling relations for systems at a critical point,[1] where there is physics at many length scales, as evidenced by divergences in perturbation theory. The renormalization group procedure consists of the iterative elimination of small length scales, whose effect on longer length scales is retained through the modification of the equations (or probability distributions) describing the physics at longer scales. It happens that the equations describing the remaining long-distance physics tend toward a definite limit as the number of iterations increases, irrespective of the details of the original equations. This is how universality is explained. The limiting equations which are obtained as the number of iterations goes to infinity (the fixed point, in renormalization group parlance) can often be found approximately and, from these, the scaling exponents.

One would like to use this program to explain the universal Kolmogorov scaling law in turbulence. This is because the forced

---

[1] Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138.

[2] Present address: Physics Department, Princeton University, Princeton, New Jersey 08544.

0022-4715/93/0400-0129$07.00/0

Navier–Stokes equations, which provide the fundamental description of turbulent flows, exhibit the same kind of divergences in perturbation theory as one finds in critical phenomena, symptomatic of the physics at many scales which they describe. Many of the words which were used to describe critical phenomena can be used to describe the situation in turbulent flows, with, however, one major difference, a difference which has made it thus far impossible to realize the hope of a renormalization group justification for the Kolmogorov law. In the case of critical phenomena, the fundamental nonuniversal interactions which are put in by hand in any model act at short distances; universal behavior is found at long distances. Turbulent flows are produced by forces which act at long distances; the behavior at these scales is nonuniversal and depends on the details of the forces. The energy which is input at large scales cascades down to small scales where the universal Kolmogorov law holds, i.e., the energy spectrum is $E(k) = C_K \varepsilon^{2/3} k^{-5/3}$ (C_K is the Kolmogorov constant and ε is the energy dissipation per unit volume). At very small scales, the energy is dissipated nonuniversally.

Despite this difference between the two physical pictures, it was thought that a renormalization group analysis might yield the Kolmogorov law, in part because Forster *et al.*[2] were successful in using renormalization group techniques to derive the large-distance, long-time behavior of the nonturbulent flows produced by certain specific forces. DeDominicis and Martin[3] generalized this result to a wider range of possible forces and showed that for one of these the Kolmogorov law results. However, the universality of the 5/3 exponent remains a mystery.

This mystery will not be solved in this paper. Rather, we consider a simpler problem which for certain values of the defining parameters has a physical picture similar to that of the opening paragraph but nevertheless presents some of the same difficulties as the Navier–Stokes equations, namely the transport of passive scalars (e.g., temperature) by an incompressible flow with a specified energy spectrum. The renormalization group treatment of this problem will be considered, in the hope that by seeing the limitations and successes of the renormalization group in this simpler arena we may learn something about the turbulence problem. In particular, we calculate the time rescaling corresponding to a spatial rescaling (i.e., the analog of the critical exponent z) in an ε approximation, where ε is a parameter describing the velocity field. We compare our results with the exact results of Avellaneda and Majda[4] for certain two-dimensional flows and propose a $1/N$ expansion which may provide useful information about those values of ε for which the ε expansion is no longer valid. Following DeDominicis and Martin, we use the field-theoretic renormalization group.[5–7]

In the next section, the basic field-theory statement of the problem is given: renormalization functions are introduced and renormalization group equations presented. The third section is a summary of the exponents that may be calculated with these equations, and the final section is a discussion of these results.

2. RENORMALIZATION GROUP EQUATIONS

The transport of a passive scalar by an incompressible flow is governed by the equation

$$\left(\frac{\partial}{\partial t} - \mathbf{v}_0 \nabla^2\right) T + v_i \partial_i T = 0 \tag{1}$$

Here T is the passive scalar, $\mathbf{v}_0$ is the kinematic viscosity, and v_i is the velocity field whose correlations are given by

$$\langle v_i(k, \omega)\, v_j(-k, -\omega)\rangle = \frac{1}{(2\pi)^{d/2}} \frac{\lambda_0}{a_0} \tau_{ij}(k) k^{2-d-\varepsilon} k^{-z} \phi\left(\frac{\omega}{a_0 k^z}\right) \tag{2}$$

where

$$\tau_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2} \qquad \text{and} \qquad \phi(u) = \frac{1}{\pi(1+u^2)} \tag{3}$$

The velocity field is assumed to have Gaussian correlations. Note that $\varepsilon = 8/3$ and $z = 2/3$ yield the Kolmogorov exponent in three dimensions.

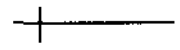
We form, in the usual way, the Martin–Siggia–Rose field theory[6] which generates the desired correlation functions. The generating functional for T correlations is

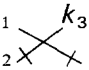
$$Z[J] = \int [dT][d\hat{T}] \exp\left(-S[T, \hat{T}] + \int d^d x\, dt\, JT\right) \tag{4}$$

where

$$S = \int d^d x\, dtt \left[\hat{T}\left(\frac{\partial}{\partial t} - \mathbf{v}_0 \nabla^2\right) T + \hat{T}\partial_i T \langle v_i v_j\rangle \hat{T} \partial_j T\right] \tag{5}$$

The Feynman rules for the bare propagator and vertex may be read off from the action (5):

 $$= \langle \hat{T} T \rangle_0 = \frac{1}{-i\omega + v_0 k^2} \tag{6}$$

$k_1; \omega_1$ $k_3; \omega_3$ $k_2; \omega_2$ 

$$\begin{aligned} &= \frac{1}{4} \frac{\lambda_0}{a_0} (k_1)_i (k_3)_j \left[\tau_{ij}(k_1 + k_2) |k_1 + k_2|^{2-d-\varepsilon-z} \phi \right. \\ &\qquad \left. \times \left(\frac{\omega_1 + \omega_2)}{a_0 |k_1 + k_2|^z} \right) + (1 \to 3) \right] \\ &\equiv M(k_1, k_2, k_3) \end{aligned} \tag{7}$$

We may now employ the usual field-theoretic renormalization group techniques.[5-7] We let the cutoff $\Lambda \to \infty$ and introduce renormalization constants for the primitive ultraviolet divergences:

$$T = Z^{1/2} T_R \tag{8}$$

$$\hat{T} = Z^{1/2} \hat{T}_R \tag{9}$$

$$\lambda_0 = \lambda \mu^\varepsilon Z_\lambda / Z^2 \tag{10}$$

$$\mathbf{v}_0 = \mathbf{v} Z_v \tag{11}$$

$$a_0 = a \mu^{2-z} Z_a \tag{12}$$

Some examples of divergent diagrams are given in Fig. 1. The renormalization constants Z, Z_λ, and Z_v are determined by the renormalization conditions at an arbitrary scale μ:

$$i \frac{\partial}{\partial \omega} \Gamma_R^{(2)}(k, \omega) \bigg|_{k^2 = \mu^2, \omega = v\mu^2} = 1 \tag{13}$$

$$\frac{\partial}{\partial k^2} \Gamma_R^{(2)}(k, \omega) \bigg|_{k^2 = \mu^2, \omega = v\mu^2} = v \tag{14}$$

$$\Gamma_k^{(4)}(k_i, \omega)|_{\mathrm{SP}} = \lambda \mu^{2-d} \left(a + \frac{1}{a} \right)^{-1} \tag{15}$$

$$\frac{\partial}{\partial \omega^2} \Gamma_R^{(4)}(k_i, \omega) \bigg|_{\mathrm{SP}} = -\frac{\lambda}{a} \mu^{z-2-d} \left(a + \frac{1}{a} \right)^{-2} \tag{16}$$

SP is the point $k_i \cdot k_j = \frac{1}{4}\mu^2(4\delta_{ij} - 1)$, $\omega = \mu^2$. Alternatively, we could just as easily use a minimal subtraction scheme.

There are now two different ways in which we can proceed, using two different renormalization group equations. One will be more convenient for $\varepsilon > 4 - 2z$ and the other for $\varepsilon < 4 - 2z$. This is because it is possible to reorganize perturbation theory so that it is better suited for the different

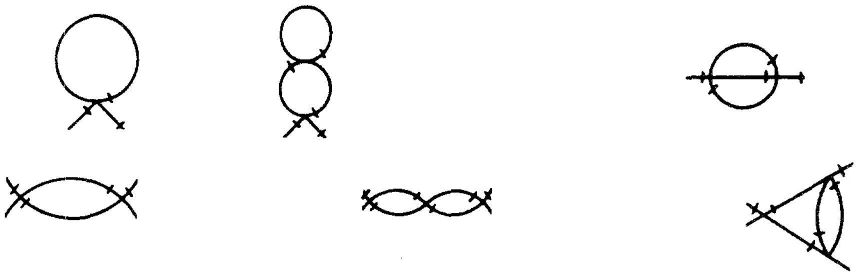


Fig. 1. Divergent diagrams.

values of ε and z: sometimes the exponent k^z will be relevant for the divergence of Feynman integrals and sometimes it will not. To develop the renormalization group equation for the former case, we write the renormalized response function $\langle \hat{T}T \rangle$ in the form

$$G(k, \omega) \equiv \langle \hat{T}T \rangle = (1/\mathbf{v})\, F(\omega/i\mathbf{v}, k, g, h, \mu) \tag{17}$$

where

$$g = \lambda \mathbf{v}^{-2} \tag{18}$$

$$h = a\mathbf{v}^{-1} \tag{19}$$

$G(k, \omega)$ satisfies a renormalization group equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_1(g, h) \frac{\partial}{\partial g} + \beta_2(g, h) \frac{\partial}{\partial h} + \eta_v - \eta + \eta_v \omega \frac{\partial}{\partial \omega}\right] G(k, \omega) = 0 \tag{20}$$

The renormalization group functions are defined by

$$\beta_1(g, h) = \mu \frac{dg}{d\mu} = -g(\eta_\lambda - 2\eta_v - 2\eta + \varepsilon) \tag{21}$$

$$\beta_2(g, h) = \mu \frac{dh}{d\mu} = -h(2 - z + \eta_a - \eta_v) \tag{22}$$

$$\eta_\lambda = \mu \frac{d}{d\mu} \ln Z_\lambda \tag{23}$$

$$\eta = \mu \frac{d}{d\mu} \ln Z \tag{24}$$

$$\eta_v = \mu \frac{d}{d\mu} \ln Z_v \tag{25}$$

$$\eta_a = \mu \frac{d}{d\mu} \ln Z_a \tag{26}$$

Solving the renormalization group equation (20) at an infrared stable fixed point, $\beta_1(g^*, h^*) = \beta_2(g^*, h^*) = 0$, we obtain the scaling form

$$G(k, \omega) \sim k^{2+\eta_v-\eta} F\left(\left[\frac{\mu}{k}\right]^{2-\eta_v} \frac{\omega}{i\mathbf{v}}, \mu, g^*, h^*, \mu\right) \tag{27}$$

Hence, the time rescaling exponent is $2 - \eta_v(g^*, h^*)$.

The alternative renormalization group equation may be derived if we write the renormalized response function $\langle \hat{T}T \rangle$ in the form

$$G(k, \omega) \equiv \langle \hat{T}T \rangle = F(\omega/a, k, u, f, \mu) \tag{28}$$

where

$$u = \lambda a^{-1} \tag{29}$$

$$f = \mathbf{v} a^{-1} \tag{30}$$

$G(k, \omega)$ satisfies a renormalization group equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_1(u, f) \frac{\partial}{\partial u} + \beta_2(u, f) \frac{\partial}{\partial f} - \eta + \eta_a \omega \frac{\partial}{\partial \omega}\right) G(k, \omega) = 0 \tag{31}$$

The renormalization group functions are defined by

$$\beta_1(u, f) = \mu \frac{du}{d\mu} = -u(\eta_\lambda - \eta_a - 2\eta + \varepsilon + z - 2) \tag{32}$$

$$\beta_2(u, f) = \mu \frac{df}{d\mu} = -f(z - 2 - \eta_a + \eta_v) \tag{33}$$

Solving the renormalization group equation (34) at an infrared stable fixed point, $\beta_1(u^*, f^*) = 0$, we obtain the scaling form

$$G(k, \omega) \sim k^{2-\eta} F\left(\left[\frac{\mu}{k}\right]^{z-\eta_a} \frac{\omega}{a}, \mu, u^*, f^*, \mu\right) \tag{34}$$

with time rescaling exponent $z - \eta_a(u^*, f^*)$.

3. CALCULATION OF EXPONENTS

We present the scaling exponents in $d = 2$ which results from the application of the above equations and briefly summarize the calculation of the requisite renormalization functions. We find four different scaling behaviors by straightforward calculation with the renormalization group equations (20) and (31). First, we will consider the three regions in which exponents may be calculated with Eq. (20):

(i.1) For $z \geqslant 2$ and $\varepsilon < 0$, the infrared stable fixed point is $g^* = h^* = 0$.[3] This is the region of mean-field theory. $\eta_v(g^* = 0) = 0$, so the time rescaling exponent is 2, the diffusive exponent.

(ii) For $z \geqslant 2$ and $\varepsilon > 0$, h is still irrelevant, so $h^* = 0$, but g is relevant, so we are interested in the fixed point $\beta_1(g^*, 0) = 0$. We may find the fixed point to any desired order in ε. To lowest order, we find scaling exponent

$$\eta_v(g^*, 0) = \varepsilon + O(\varepsilon^2) \tag{35}$$

(iii) For $z < 2$ and $\varepsilon > 4 - 2z > 0$, both g and h are relevant, and we calculate the exponents as a double expansion in ε and $2 - z$. To lowest order in both, we have scaling exponent

$$\eta_v(g^*, h^*) = \varepsilon - (2 - z) + O(\varepsilon^2, (2 - z)^2) \tag{36}$$

We run into trouble, however, once we consider $\varepsilon < 4 - 2z$. To see why, observe that when $\beta_2(g, h) = 0$, we may write

$$\beta_1(g, h) = -g(\eta_\lambda - 2\eta_a - 2\eta + \varepsilon - 4 + 2z) \tag{37}$$

For $\varepsilon < 4 - 2z$, $g^* = 0$ is a stable fixed point, so the fixed point which we used for case (iii) is not. Unfortunately, we must find a nonzero fixed point, since, for $z < 2$,

$$\beta_2(0, h) = -h(2 - z) \tag{38}$$

so there is no infrared stable fixed point of $\beta_2(0, h)$. Hence, we need to find some other fixed point of (37) to proceed. Fortunately, we may, instead, use the renormalization group equation (31). We find the following two behaviors:

(i.2) For $z < 2$ and $\varepsilon < 2 - z$, both u and f are irrelevant, so we have mean-field behavior.

(iv) For $z < 2$ and $2 - z < \varepsilon < 4 - 2z$, u is relevant, but f is still irrelevant.[4] We may calculate the scaling exponents as a power series in ε and $2 - z$. To lowest order, we find

$$\eta_a(u^*, 0) = \varepsilon - 2(2 - z) + O(\varepsilon^2, (2 - z)^2) \tag{39}$$

[3] To see that this fixed point is infrared stable, observe that η_λ, η_v, and η are $O(g)$, so $\beta'(g = 0) = -\varepsilon$. Hence $g^* = 0$ is an infrared stable fixed point for $\varepsilon < 0$ but unstable for $\varepsilon > 0$, just as in the case of ϕ^4 theory and critical phenomena.

[4] The upper bound on ε is necessary in order for f to be irrelevant.

or, equivalently, in terms of the other parameters,

$$\eta_a(g^*, h^*) = \varepsilon + z - 2 \tag{40}$$

At the boundaries between these regions [except at that between (iii) and (iv)], there are marginal operators, so logarithmic corrections are expected. These are straightforward to calculate since they are of the type that one finds in four-dimensional quantum field theory.

To derive the preceding results (i)–(iv), it is sufficient to find the RG functions η_λ, η_a, η, and η_v to $O(g^2)$ [for the results (i.2) and (iv), we need to reshuffle the expansion so that it is in terms of u and f]. The relevant diagrams are displayed in Fig. 1. Using the normalization conditions (13)–(16), we obtain

$$Z = 1 + \frac{1}{16} i g^2 \frac{\partial}{\partial\omega} I(k, \omega)\bigg|_{k^2 = \mu^2, \omega = v\mu^2} + O(g^3) \tag{41}$$

$$ZZ_v = 1 + \frac{\partial}{\partial k^2}\left\{\frac{1}{4} g I_v(k) - \frac{1}{16} g^2 [I_v(k)]^2 - \frac{1}{16} g^2 I(k, \omega)\right\}\bigg|_{k^2 = \mu^2, \omega = v\mu^2} + O(g^3) \tag{42}$$

$$Z_\lambda - Z_a^{-2} - Z_\lambda Z_a^{-2} = 1 + \frac{1}{4} g I_\lambda(\mu) + \frac{1}{16} g^2 [I_\lambda(\mu)]^2 + \frac{1}{16} g^2 I'_\lambda(\mu) + O(g^3) \tag{43}$$

$$\begin{aligned}&(Z_\lambda - Z_a^{-2} - Z_\lambda Z_a^{-2})(2 + a^{-2} Z_a^{-2} - a^2 Z_a^2)\\ &\quad = 1 + \frac{\partial}{\partial\omega^2}\left\{\frac{1}{4} g I_\lambda(\mu) + \frac{1}{16} g^2 [I_\lambda(\mu)))]^2 + \frac{1}{16} g^2 I'_\lambda(\mu)\right\}\bigg|_{\mathrm{SP}} + O(g^3)\end{aligned} \tag{44}$$

where the Feynman integrals I, I_v, I_λ, and I'_λ are discussed in the Appendix. The desired renormalization functions may be calculated using (23)–(26). This is done most easily by differentiating the following equations with respect to μ, using the chain rule, and solving for β_1 and β_2 (the left-hand sides vanish upon differentiation) (see, e.g., ref. 6, Chapter 9):

$$g_0 = g\mu Z_\lambda Z^{-2} Z_v^{-2} \tag{45}$$

$$h_0 = h\mu^{2-z} A_a Z^2 \tag{46}$$

Then η_i $(i = \mathbf{v}, a)$ may be calculated according to

$$\eta_i = \beta_1 \frac{dc}{dg} Z_i + \beta_2 \frac{d}{dh} Z_i \tag{47}$$

4. CONCLUSION

Let us stop now to make a few observations. First, let us compare our results with the exact results of Avellaneda and Majda[4] for a two-dimensional shear flow (i.e., the velocity field is directed along the y axis and is independent of y). They find five different behaviors. For $z > 2$ they find $\eta_\nu(g^*) = 0$ for $\varepsilon < 0$, and for $0 < \varepsilon < 2$ they find $\eta_\nu(g^*) = \varepsilon/(1 + \varepsilon/2)$. For $z < 2$ they find mean field behavior for $\varepsilon < 2 - z$, $\eta_\nu(g^*) = \varepsilon + z - 2$ for $2 - z < \varepsilon < 4 - 2z$, and $\eta_\nu(g^*) = (2z - 4 + 2\varepsilon)/(2z - 2 + \varepsilon)$ for $4 - 2z < \varepsilon < 2$. These are the behaviors (i)–(iv) which our analysis has reproduced for small ε. However, for $\varepsilon > \max(2, 4 - 2z)$, they find $\eta_\nu(g^*) = \varepsilon/2$. Hence, our results, which are good for small ε, are completely wrong for $\varepsilon > \max(2, 4 - 2z)$. The above results for cases (i)–(iii) have also been obtained[8] to lowest order in ε using the renormalization group techniques of Yakhot and Orszag[9] (which are more in the spirit of Wilson's original presentation). However, the analysis in ref. 8 does not yield the correct behavior for case (iv). This is essentially because the renormalization group techniques used in ref. 8 correspond to the renormalization group equation (20), rather than Eq. (31), which allows one to find the correct fixed point at $O(\varepsilon, (2 - z))$ for case (iv).

The region $\varepsilon > \max(2, 4 - 2z)$, where our analysis fails, is of great interest for several reasons. The Kolmogorov spectrum ($\varepsilon = 8/3$, $z = 2/3$) occurs on the boundary of the region. Furthermore, this region is similar to the region of interest for the Navier–Stokes equations since, presumably, this new behavior is due to the appearance of a new relevant operator—which is precisely what happens at the boundary of the infrared pumping region of the forced Navier–Stokes equations.[3] There is one major difference, however. The external parameters, i.e., the exponents describing the random force, of the Navier–Stokes equations determine the naive operator dimensionality of the basic fields; thus, it is easy to see the appearance of new relevant operators as these parameters are varied. In the case of transport, however, this is not the case. Only g and h have their naive dimensions determined by the values of ε and z. Variation of ε and z can affect only the anomalous dimensions of other operators. This makes it harder to find new relevant operators since their relevance must be due to their anomalous dimensions. However, this is also an attractive feature since it implies that the scaling in the region $\varepsilon > \max(2, 4 - 2z)$ is not written in by hand via the naive operator dimensionality but is determined by the dynamics of the underlying equations—which is the type of universality that we would like.

If we wish to explore by general means (i.e., not special to two dimensions) the regions which are beyond the reach of the ε expansion, some other approximation is needed to reproduce the results of ref. 4; presumably, such an approximation will be useful for transport in $d > 2$ where exact results are not available. One hope is that a $1/N$ approximation will provide some information about this region. This approximation is developed in the following way. We consider N passive scalars and take coupling constant λ/N. Several simplifications result if we consider only the lowest-order diagrams in $1/N$. First, $\Gamma^{(4)}$ becomes calculable as a power series in g. Second, the only diagrams which contribute to $\Gamma^{(2)}$ are the tadpole insertions in the propagator. Hence, $Z = 1$. The only RG function that cannot be calculated to infinite order in g is Z_ν. The diagrammatics is the same as that of ϕ^4 theory, the only difference between the two theories being that in the ϕ^4 theory tadpole insertions merely renormalize the critical temperature, while in this case they are physically interesting because they determine Z_ν. To be more specific, we begin with N passive scalars,

$$\left(\frac{\partial}{\partial t} - \mathbf{v}_0 \nabla^2\right) T^a + v_i \partial_i T^a = 0 \tag{48}$$

($a = 1, 2, \ldots, N$) and form the generating functional. Replacing λ by λ/N, we arrive at the action

$$S = \int d^dx\, dt \left[\hat{T}^a \left(\frac{\partial}{\partial t} - \mathbf{v}_0 \nabla^2\right) T^a + \frac{1}{N} \hat{T}^a\, \partial_i Ta \langle v_i v_j \rangle \hat{T}^b\, \partial_j T^b \right] \tag{49}$$

The RG functions may be calculated to lowest nonvanishing order in $1/N$:

$$Z = 1 \tag{50}$$

$$Z_\lambda - Z_a^{-2} - Z_\lambda Z_a^{-2} = \left[1 - \frac{1}{4} g I_\lambda(\mu)\right]^{-1} \tag{51}$$

$$Z_\nu = 1 + \frac{\partial}{\partial k^2} \left\{ \frac{1}{4} g I_\nu(k)) - \frac{1}{16} g^2 [I_\nu(k)]^2 \right\} \Bigg|_{k^2 = \mu^2,\, \omega = \nu\mu^2} + O(g^3) \tag{52}$$

Unfortunately, some further approximation must be made in order to evaluate Z_ν. Hence, we do not learn anything directly about the region of interest. However, we have pinpointed the diagrams which cause the most difficulty. A method for dealing effectively with these diagrams—perhaps within the $1/N$ expansion—is needed for further progress. It is possible that these approximate renormalization groups which must be developed for the transport problem may shed new light on the Navier–Stokes equations and turbulence, since they are intended for a region of the former which is similar to the region of greatest interest of the latter.

APPENDIX. FEYNMAN INTEGRAL CALCULATION

The Feynman integrals I, I_v, and I_λ corresponding to the diagrams in Fig. 1 are

$$I = \frac{\mu^{2\varepsilon}}{(2\pi)^{2d}} \int d^d k_1\, d^d k_2\, d\omega_1\, d\omega_2 \times \frac{M(-k_2, -k_1, k)\, M(k_1, k_2, k-k_1-k_2)}{(-i\omega_1/\mathbf{v} + k_1^2)(-i\omega_2/\mathbf{v} + k_2^2)[-i(\omega-\omega_1-\omega_2)/\mathbf{v} + (k = k_1 - k_2)^2]} \tag{A1}$$

$$I_v = \frac{\mu^{\varepsilon}}{(2\pi)^d} \int d^d k_1\, d\omega_1 \frac{M(-k_1, k_1, k)}{-i\omega_1/\mathbf{v} + kk_1^2} \tag{A2}$$

$$I_\lambda = \frac{\mu^{2\varepsilon}}{(2\pi)^d} \int d^d k\, d\omega \frac{M(k_3, -k_1-k_2-k_3, k)\, M(k_1, k_2, k-k_1-k_2)}{(-i\omega/\mathbf{v} + k^2)[-i(\omega_1+\omega_2-\omega)/\mathbf{v} + (k-k_1-k_2)^2]} \tag{A3}$$

$$I'_\lambda = \frac{\mu^{2\varepsilon}}{(2\pi)^{2d}} \int d^d k\, d^d k'\, d\omega\, d\omega' \times \frac{M(-k_1-k_2-k_3, k, k')\, M(k_1, k_2, -k)}{(-i\omega'/\mathbf{v} + k'^2)(-i\omega/\mathbf{v} + k^2)[-i(\omega_1+\omega_2-\omega)/\mathbf{v} + (k-k_1-k_2)^2]} \times \frac{M(-k', k_3, k_1+k_2+k_3-k-k')}{[-i(\omega_1+\omega_2+\omega_3-\omega-\omega')/\mathbf{v} + (k_1+k_2+k_3-k-k')^2]} \tag{A4}$$

These integrals may be evaluated through the use of Feynman parameters. As an example, we present the evaluation of I_v for $z \geqslant 2$ (we set $h = 0$, its fixed-point value for $z \geqslant 2$). The ω integral may be done immediately by contour integration. Consider, then, the two terms which contribute to I_v:

$$\mu^{-\varepsilon} I_v = \int \frac{d^d k_1}{(2\pi)^d} \frac{k \cdot k_1}{k_1^2 |k_1 + k|^{\varepsilon+d-2}} + \int \frac{d^d k_1}{(2\pi)^d} \frac{(k_1^2 + k \cdot k_1)(k^2 + k \cdot k_1)}{k_1^2 (k_1 + k)^2 |k_1 + k|^{\varepsilon+d-2}} \tag{A5}$$

The denominators may be combined using Feynman parameters and the integrals shifted:

$$\mu^{-\varepsilon} I_v = -\iint \frac{d^d k_1}{(2\pi)^d} dx \frac{k^2 (1-x)^{(\varepsilon+d)/2-1}}{[k_1^2 + (x-x^2)k^2]^{(\varepsilon+d)/2}} + \iint \frac{d^d k_1}{(2\pi)^d} \frac{[(d+2)x-1]k_1^2 k^2 + dk^4}{d[k_1^2 + (x-x^2)k^2]^{(\varepsilon+d)/2+1}} \tag{A6}$$

Performing the k_1 integrals, we obtain

$$\mu^{-\varepsilon} I_v = -\frac{k^{2-\varepsilon}}{(4\pi)^{d/2}} \left\{ \frac{\Gamma(\varepsilon/2)}{\Gamma((\varepsilon+d-2)/2)} \int dx\, x^{-\varepsilon/2}(1-x)^{d/2-2} \right.$$
$$+ \frac{\Gamma((\varepsilon+2)/2)}{\Gamma((\varepsilon+d+2)/2)} \int dx\, x^{-\varepsilon/2-1}(1-x)^{d/2-2}$$
$$\left. + \frac{\Gamma(\varepsilon/2)}{\Gamma((\varepsilon+d)/2)} \int dx\, [(d+2)x^{1-\varepsilon/2} - x^{-\varepsilon/2}](1-x)^{d/2} \right\} \tag{A7}$$

Using the identity

$$\int dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{A8}$$

we may perform the x integrals to obtain

$$\mu^{-\varepsilon} I_v = -\frac{k^{2-\varepsilon}}{(4\pi)^{d/2}} \left\{ \frac{\Gamma(\varepsilon/2)\,\Gamma(1-\varepsilon/2)\,\Gamma(d/2-1)}{\Gamma((\varepsilon+d-2)/2)\,\Gamma((d-\varepsilon)/2)} \right.$$
$$+ \frac{\Gamma((\varepsilon+2)/2)\,\Gamma(-\varepsilon/2)\,\Gamma(d/2-1)}{\Gamma((\varepsilon+d+2)/2)\,\Gamma((d-\varepsilon)/2-1)}$$
$$\left. + \frac{\Gamma(\varepsilon/2)\,\Gamma(d/2+1)}{\Gamma((\varepsilon+d)/2)} \left[\frac{\Gamma(2-\varepsilon/2)}{\Gamma((d-\varepsilon)/2+3)} - \frac{\Gamma(1-\varepsilon/2)}{\Gamma((d-\varepsilon)/2+2)} \right] \right\} \tag{A9}$$

So, for instance, we can calculate to lowest order in ε

$$\left.\frac{\partial}{\partial k^2} I_v(k)\right|_{k^2=\mu^2} = \frac{\varepsilon}{2} \frac{1}{(4\pi)^{d/2}} \left\{ \frac{\Gamma(d/2-1)}{\Gamma((d-2)\,\Gamma(d/2)} - \frac{\Gamma(d/2-1)}{\Gamma((d+2)/2)\,\Gamma(d/2-1)} \right.$$
$$\left. + \frac{\Gamma(d/2+1)}{\Gamma(d/2)\,\Gamma(d/2+3)} - \frac{\Gamma(d/2+1)}{\Gamma(d/2)\,\Gamma(d/2+2)} \right\} \tag{A10}$$

For $z<2$, we must expand the integral in powers of $(z-2)$. For example, we write

$$\frac{1}{hk^z + k^2} = \frac{1}{2k^2} \left[1 - \frac{1}{4} h(z-2) \ln k^2 + \cdots \right] \tag{A11}$$

and proceed as above.

ACKNOWLEDGMENT

I would like to thank Prof. Paul C. Martin for many long and helpful discussions. Without him this work would have been neither conceived nor completed.

REFERENCES

1. K. G. Wilson and J. Kogut, *Phys. Rep.* **12C**:75 (1974).
2. D. Forster, D. Nelson, and M. Stephens, *Phys. Rev. A* **16**:732 (1977).
3. C. DeDominicis and P. C. Martin, *Phys. Rev. A* **19**:419 (1979).
4. M. Avellaneda an d A. Majda, *Commun. Math. Phys.* **131**:381 (1990).
5. E. Brezin, J. C. LeGuillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena*, Vol. VI, C. Domb and M. S. Green, eds. (Academic, New York, 1975).
6. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, 1989).
7. C. DeDominicis and L. Peliti, *Phys. Rev. B* **18**:353 (1978).
8. M. Avellaneda and A. Majda, *Phys. Fluids A* **4**:41 (1992).
9. V. Yakhot and S. Orszag, *J. Sci. Comp.* **1**:3 (1986).